by Ian Stewart

Cone with a Twist

The cone is probably most familiar today as an edible container for ice cream, but its past glories lie in higher realms altogether. The geometry of the cone was known to the ancient Greeks, mainly because of the elegant curves that could be constructed by slicing a cone with a plane. The Greeks delighted in the intricate geometry of these conic sections – the ellipse, parabola and hyperbola – and discovered how to use them to solve problems that were beyond the reach of rulers and compasses.

Those problems included trisecting an angle and constructing a cube with twice the volume of a given cube. Both reduce to solving an equation of the third degree—that is, one in which the highest exponent of the variable is three. Conic sections can be used to solve the problems because the points where two sections meet correspond to solutions of equations of the third and fourth degrees. In contrast, rulers and compasses can solve only second-degree equations.

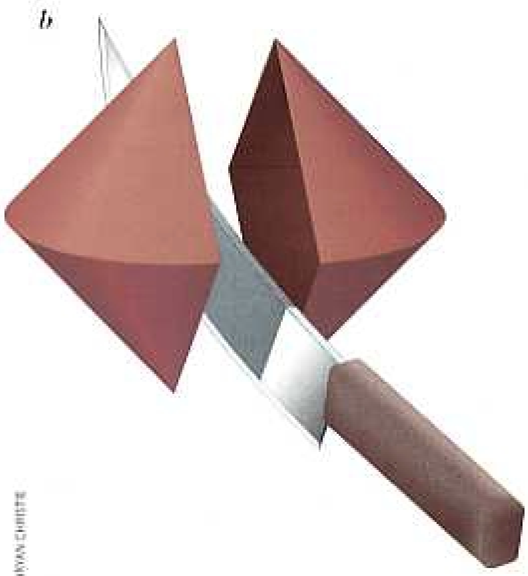
The cone itself has generally been of less interest to mathematicians than its planar sections, perhaps because the cone is so simple in form. Is there anything new to say about this humble solid? Indeed there is. In May 1999 C. J. Roberts, a reader of this column who lives in the English town of Baldock, wrote to me about a very curious shape that he calls a sphericon. He even enclosed two of them in his letter.

The sphericon [*see illustration below*] is a double-cone—two identical cones joined base to base—but with a twist. It is easy to make. If you slice a double-cone along a plane that includes both vertices, you get a rhombic cross section, a parallelogram with four equal sides. If you use cones of just the right shape, you get a square cross section. Unlike all other rhombuses, the square has an extra symmetry: rotate it through a right angle and it fits back into the same shape. So you can slice such a double-cone down the middle, rotate one of the halves through a right angle and glue the two pieces back together. This is the sphericon. Thanks to the twist, it is not a double-cone but a much more interesting beast. I'd never seen such a shape mentioned anywhere before, but if I'm wrong about its novelty I'm sure that a better-informed reader will quickly tell me.

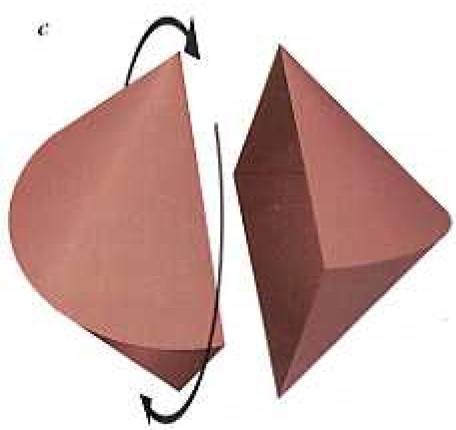
The sphericon can be made from a single piece of thin card, cut to a shape made from four identical sectors of a circle joined together so that they face in alternate directions [*see illustration on opposite page*]. The main calculation involved in designing this shape is to find the angle between the two straight edges of the sector. Suppose the length of each edge is 1. If the double-cone has a square cross section, the Pythagorean theorem says the base of each component cone has a diameter of $\sqrt{2}$. So the circumference of the base is $\pi\sqrt{2}$. The length of the sector's arc is half that (because you cut the double-cone in half to make a sphericon). The angle of the sector therefore works out to be $\pi\sqrt{2}/2$ radians, or $90\sqrt{2}$ degrees, which is approximately 127.28 degrees.

a

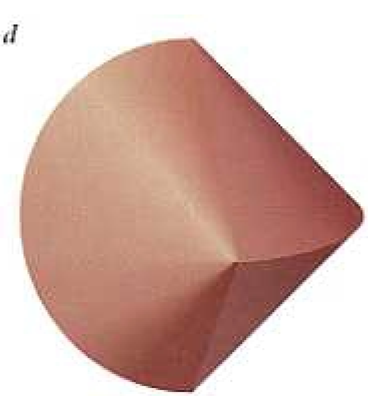
b



c

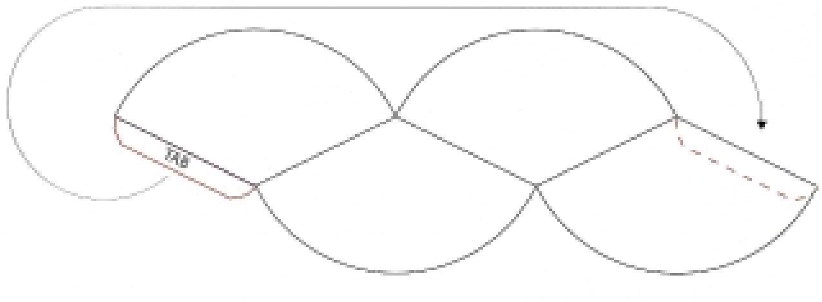


d



TO CONSTRUCT A SPHERICON join two cones at the base (a), then cut along the plane that includes the vertices (b). Rotate one half 90 degrees (c), then glue the halves together (d).

BRYAN CHRISTIE



AN EASIER WAY TO BUILD A SPHERICON
is to cut a piece of cardboard in the shape above (but enlarged), glue the tab to the matching edge and join the bases of the half-cones.

If you cut out the shape shown in the illustration, you can roll up the sectors into half-cones and glue the tab to the matching edge. With a little adjustment the bases of the half-cones will fit snugly with no gap, and you can tape the joins.

The first delight of the sphericon is: it rolls! A cone placed on a flat surface rolls around in circles. A double-cone can roll in a clockwise circle or a counterclockwise one, but it rolls straight only if you rapidly bowl it or set it on rails. A sphericon performs a controlled wiggle, which on average is straight. First one conical sector is in contact with the flat surface, then the next. So as it moves forward it wiggles alternately to the left and right. It is especially interesting to start it at the top of a gradual slope and watch it wobble its way down. After reading the letter from Roberts, I spent a pleasant half-hour with several other mathematicians rolling sphericons along a table.

The letter also hinted at some of the sphericon's fascinating attributes: it has one continuous face, and one sphericon can roll around another ad infinitum. Intrigued, I asked for more information, and in return Roberts sent an enormous cardboard box that weighed virtually nothing. It contained a large lattice of about 50 sphericons, neatly assembled with transparent tape. This lattice, like the atomic lattice of a crystal, repeats indefinitely in three dimensions.

One reason why the sphericon has such neat geometric properties is that its four "edges"–the lines where the component sectors meet–lie along four of the edges of a regular octahedron. The other four edges of the octahedron correspond to lines that bisect the vertex angles of the sectors. The octahedron, in turn, is closely related to the cube: if you put a dot in the middle of each face of a cube and join the dots by straight lines, you get an octahedron. And cubes, of course, stack in a regular manner to form a flat layer or fill three-dimensional space.

Roberts, who is 47, invented the sphericon about 30 years ago. Geometry was his strong point at school, and he started work as a joiner's apprentice. Not surprisingly, therefore, his first sphericon was carved out of wood. His starting point was the well-known Möbius strip, a band of paper joined end to end with a 180-degree twist. Roberts realized that because paper has a definite thickness, the band's cross section is really a long, thin rectangle. If you make the cross section into a square, you can join the ends with a 90-degree twist instead, producing a solid whose outer surface consists of a single curved face. This shape, however, has a hole in the middle: it is a ring. Does there exist a solid that is not a ring whose outside has a single curved face? One day, while Roberts was working on a length of wood with a square cross section, he started thinking about blending one face into the next by planing a curve around the ends. Do this at both ends, eliminate the wood in between, and you get a sphericon.

He made one out of mahogany and gave it to his sister, who has kept it ever since. Then he forgot the topic until 1997, when I gave a series of televised mathematics lectures and talked about symmetry. At that point Roberts's interest was revived, and he wrote to me.

If two sphericons are placed next to each other, they can roll on each other's surfaces. Four sphericons arranged in a square block can all roll around one another simultaneously. And eight sphericons can fit on the surface of one sphericon so that any one of the outer solids can roll on the surface of the central one.

The possible arrangements of sphericons seem endless. I leave to readers the pleasure of playing with this extremely clever mathematical toy and inventing new patterns for themselves. SA